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Analytical VaR for international portfolios with common jumps

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ABSTRACT

International portfolios which are composed of domestic assets and foreign assets are popular investment tools for financial institutions in highly integrated global financial markets. However, the focus of past studies had been on either domestic assets or foreign assets, but not both in the same context. They paid no attention to the studies of controlling the market risk of the international portfolios in the risk management literature. In contrast to the existing literature in portfolios, this paper considers not only domestic assets but also foreign assets, and provides an analytical value-at-risk (VaR) with common jump risk and exchange rate risk to manage market risk of international portfolios with exchange rate risk and common jumps over the subprime mortgage crisis. In general, the analytical solution can be used to accurately calculate VaRs by the backtesting criterion in terms of in-sample and out-of-sample fitting for an international portfolio with common jumps.

1. Introduction

Nowadays, the investment in foreign currency assets circulates rapidly around the world. In Taiwan, the official monthly statistic reports offered by the Central Bank of Taiwan illustrate that the average percentage of investment in foreign assets relative to domestic assets has been approximately 46% at domestic commercial banks over the past ten years. In Japan, the ratio is at least 5%, and in Korea it is around 9%. On average, the percentage of overall portfolio allocation to foreign assets is around 20% at Asian banks, and the percentage is growing. Thus, controlling the market risk of portfolios composed of domestic assets and foreign assets is an increasing concern for financial institutions.

The VaR approach is a popular tool to manage market risk, which is defined as the maximum loss over a fixed target horizon with a given probability. Using the VaR measure, Hofmann and Platen [1] consider the market risk of a large diversified portfolio in which the dynamic process of asset returns is distributed in normal diffusion. Equally, the asset price follows a lognormal distribution. However, substantial evidence exists in the empirical financial economic literature of the existence of jumps in equity returns and foreign exchange rates such as [2–4]. Therefore, the lognormal assumption is, in actuality, contrary to real life. Daily changes in many variables, especially in exchange rates, illustrate significant positive kurtosis. This means that the probability distributions of asset returns have fat tails or discontinuity. Literature related to these studies has been presented by Stock and Watson [5], Hull and White [6], Hansen [7], and Consigli [8]. Besides them, Shang et al. [9] employ a jump–diffusion process to price catastrophe mortality bonds; Liu et al. [10] consider a class of stochastic optimal parameter selection problems described by linear stochastic differential equations with jumps to show that the constrained stochastic impulsive optimal parameter selection problem is equivalent to a deterministic impulsive optimal parameter; Ma and Zhao [11] and Tin et al. [12] apply a jump–diffusion process to a simulation analysis of nearest-neighbour rule under stochastic demand and a web reliability ranking system. Alternatively, Gibson [13] demonstrates that event risk poses large jumps to fat tails in market prices, and incorporates event risk into VaR for a portfolio. Differing from

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the assumption held by Hofmann and Platen [1] and Guan et al. [14], Gibson considers jump–diffusion asset returns to model large diversified portfolios. As stated above, the literature focuses on the portfolios only valued in one currency. However, it is a common phenomenon for institutional and individual investors to invest in the portfolios which include a number of domestic-valued assets and foreign-valued assets in highly integrated global financial markets, called international portfolios. Therefore, exchange rate risk should be considered in highly international investment.

This paper aims to present an analytical VaR formula for international portfolios. Using the framework provided by Merton [15], we employ return jumps at Poisson arrivals to avoid the assumption of normality of asset returns. Also, the Brownian motions of between-jump returns are correlated. In general, the model solution is more accurate than that of the Monte Carlo simulation techniques which are often adopted in fat-tail distributions in terms of the system infrastructure and computation time. In addition, this model can be also applied to large portfolios. Compared with that of Hofmann and Platen [1] and Guan et al. [14], the proposed model considers not only jumps but also exchange rate risk. It is more suitable to fit to real situations in highly integrated global financial markets.

The rest of this paper is organized as follows. The next section outlines the model, and an analytic formula of the value at risk is derived. In the Section 3, we first employ an international portfolio including domestic assets and foreign assets to estimate model parameters. Then, the one-day VaRs at 99% significance level for the international portfolio are calculated, and a comparative static analysis on the risk capital is provided. Using the usual backtesting criterion, Section 4 inspects the model accuracy in terms of in-sample and out-of-sample fitting over the subprime mortgage crisis of August 2007. The samples in this study span from January 1, 2004 to November 27, 2009, or 1367 daily log returns of a line of domestic assets and foreign assets. The last section provides conclusions.

2. Model formulation

First, this paper assumes that (i) a value of an international portfolio is made up of the value of n_d kinds of domestic assets with $m_{i,t}$ shares and n_f classes of foreign assets with $g_{i,t}$ shares for each $i \in 1, 2, \ldots, n$; (ii) the capital market is a complete market with no transaction cost or tax; (iii) there exists a riskless interest rate for lenders and borrowers; (iv) the dynamic processes of domestic asset returns, foreign asset returns and exchange rate returns follow Poisson jump–diffusion over the interval of interest; (v) exchange rates are quoted at the price of one unit of the foreign currencies in domestic dollars, and (vi) investment strategies do not vary over an investment horizon. The dynamic processes of asset price and exchange rates are demonstrated as follows, respectively.

$$\frac{\mathrm{d}A_{d_i,t}}{A_{d_i,t}} = (\mu_{d_i} - \lambda\upsilon)\mathrm{d}t + \sigma_{d_i}\mathrm{d}W_{1,t} + (\pi - 1)\mathrm{d}Y_t, \tag{1}$$

$$\frac{\mathrm{d}A_{f_i,t}}{A_{f_i,t}} = (\mu_{f_i} - \lambda\upsilon)\mathrm{d}t + \sigma_{f_i}\mathrm{d}W_{2,t} + (\pi - 1)\mathrm{d}Y_t, \tag{2}$$

$$\frac{\mathrm{d}e_{i,t}}{e_{i,t}} = (\mu_{e_i} - \lambda\upsilon)\mathrm{d}t + \sigma_{e_i}\mathrm{d}W_{3,t} + (\pi - 1)\mathrm{d}Y_t, \tag{3}$$

where μ_{d_i}, μ_{f_i}, and μ_{e_i} denote the constant drift rates of domestic asset returns, foreign asset returns and exchange rate returns for each $i \in 1, 2, \ldots, n$, respectively; σ_{d_i}, σ_{f_i}, and $\sigma_{e_i,t}$ stand for the constant volatilities of domestic asset returns, foreign asset returns and exchange rate returns for each $i \in 1, 2, \ldots, n$, respectively. The $W_{j,t}$ are one dimensional standard Brownian motions under the original probability measure, P for all $j = 1, 2, 3$. Also, the correlation coefficients among the three Brownian motions are defined as $\mathrm{corr}(\mathrm{d}W_{1,t}, \mathrm{d}W_{2,t}) = \rho_{1,2}$, $\mathrm{corr}(\mathrm{d}W_{2,t}, \mathrm{d}W_{3,t}) = \rho_{2,3}$, and $\mathrm{corr}(\mathrm{d}W_{1,t}, \mathrm{d}W_{3,t}) = \rho_{1,3}$.[1] Then, Y_t is an independent Poisson process with the intensity λ at time t; $\mathrm{d}Y_t$ is independent of $\mathrm{d}W_{j,t}$ for all $j = 1, 2, 3$. The υ represents $E[\pi - 1]$ where $\pi - 1$ is the random variable percentage in domestic assets or exchange rates resulting from a jump, and $E(.)$ is the symbol of the expectation operator over the random variable Y_t. Assume that the nature logarithm of π, which is the jump amplitude if Poisson events occur, follows normal distributions with the mean u_π and variance σ_π^2. That is also denoted as $\ln \pi \sim N(u_\pi, \sigma_\pi^2)$, and $\upsilon = E[\pi - 1] = \exp\left[u_\pi + \frac{1}{2}\sigma_\pi^2\right] - 1$.

Now, consider the potential daily loss exposure to long trading positions. Typically, the VaR is a specific left-hand critical value of a potential loss distribution. Given conventions, one can define the daily losses valued in domestic dollars relative to the end-of-period expected asset value (relative VaR) and the initial asset value (absolute VaR), denoted by VaR(*mean*) and VaR(0) as follows, respectively:

$$\begin{aligned} &\mathrm{VaR}(\mathit{mean}) \equiv V_\alpha - E_t(V_T), \\ &\mathrm{VaR}(0) \equiv V_\alpha - V_0, \end{aligned} \tag{4}$$

[1] For simplicity, we assume that the dependence structure between exchange rates and equity returns is linear. However, there are some drawbacks. First, it is not invariant to transformations of the original variables. Second, conditional correlations are not accounted for. Third, the proposed method cannot be used in the case of portfolios that include assets with non-linear payoffs.

in which the $E_t(.)$ is the expected value conditional on information at time t, the V_α is the value of an international portfolio denominated in domestic dollars given a percentile of α, and V_T is the portfolio value at time T (investment horizon). The international portfolio consists of n_d kinds of domestic assets and n_f kinds of foreign assets, denoted as $V_T = \sum_{i=1}^{n_d} m_i A_{d_i,T} + \sum_{i=1}^{n_f} g_i e_{i,T} A_{f_i,T}$. Which definition of value at risk provides a more suitable measure of risk capital allocation over investment horizon? Kupiec [16, page 43] demonstrates that the absolute VaR is a more appropriate measure of an asset's risk of posting losses. Thus, we adopt the measure throughout this article.

Before the derivation of the VaR analytic formula for an international portfolio, it is necessary to employ the following propositions.

Proposition 1. *Given the dynamic processes of foreign currency denominated asset price and exchange rate following the Geometric Brownian motion, the dynamic process of* $\sum_{i=1}^{n_f} g_i e_{i,t} A_{f_i,t}$ *can be expressed as*

$$\frac{\mathrm{d}X_{i,t}}{X_{i,t}} = \sum_{i=1}^{n_f} g_i \left[(\mu_{f_i} + \mu_{e_i} - 2\lambda v + \rho_{2,3}\sigma_{f_i}\sigma_{e_i})\mathrm{d}t + \sigma_{f_i}\mathrm{d}W_{2,t} + \sigma_{e_i}\mathrm{d}W_{3,t} + 2(\pi - 1)\mathrm{d}Y_t \right]$$

with $X_{i,t} = \sum_{i=1}^{n_f} g_i e_{i,t} A_{f_i,t}$.

Appendix A provides a detailed proof of Proposition 1.

Proposition 2. *Given the dynamic processes of asset price and exchange rates, the dynamic process of* V_t *can be expressed as*

$$\begin{aligned}\frac{\mathrm{d}V_t}{V_t} &= \left[\sum_{i=1}^{n_d} \gamma_{i,t}(\mu_{d_i} - \lambda v) + \sum_{i=1}^{n_f} \beta_{i,t}(\mu_{e_i} + \mu_{f_i} - 2\lambda v + \rho_{2,3}\sigma_{f_i}\sigma_{e_i}) \right] \mathrm{d}t \\ &\quad + \sum_{i=1}^{n_d} \gamma_{i,t}\sigma_{d_i}\mathrm{d}W_{1,t} + \left[\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{f_i} \right] \mathrm{d}W_{2,t} + \left[\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{e_i} \right] \mathrm{d}W_{3,t} + 2(\pi - 1)\mathrm{d}Y_t\end{aligned}$$

with $V_t = \sum_{i=1}^{n_d} m_i A_{d_i,t} + \sum_{i=1}^{n_f} g_i e_{i,t} A_{f_i,t}$, $\gamma_{i,t} = \frac{m_i A_{d_i,t}}{V_t}$, *and* $\beta_{i,t} = \frac{g_i e_{i,t} A_{f_i,t}}{V_t}$. $\gamma_{i,t}$ *and* $\beta_{i,t}$ *are also named the weights (percentage) of the investment in the ith kind of domestic asset and in the ith kind of foreign asset, respectively.*

Appendix B provides a detailed proof of Proposition 2. Conditional on assumption (vi), the weights can be regarded as constant over the investment horizon.

Using the previous propositions, one can quickly obtain the approximation of the absolute VaR by utilizing the analytic formula below:

$$\sum_{k=0}^{\infty} \frac{\exp[-\lambda T]\,[\lambda T]^k}{k!} \Phi \left(\frac{\ln[V_0 + \mathrm{VaR}(0)] - \ln V_0 - \left(\mu_t - \frac{1}{2}\sigma_t^2\right) T - k u_\xi}{\sqrt{\sigma_t^2 T + k\sigma_\xi^2}} \right) = \alpha, \tag{5}$$

in which the Z_α stands for a critical value with a given probability α; V_0 represents the initial value of an international portfolio; $v = E[\pi - 1] = \exp\left[u_\pi + \frac{1}{2}\sigma_\pi^2\right] - 1$; u_ξ and σ_ξ^2 denote the expected value and the variance of the nature logarithm of $2\pi - 1$, respectively;

$$\begin{aligned}\mu_t &= \sum_{i=1}^{n_d} \gamma_{i,t}(\mu_{d_i} - \lambda v) + \sum_{i=1}^{n_f} \beta_{i,t}(\mu_{e_i} + \mu_{f_i} - 2\lambda v + \rho_{2,3}\sigma_{f_i}\sigma_{e_i});\\ \sigma_t^2 &= \left(\sum_{i=1}^{n_d} \gamma_{i,t}\sigma_{d_i} \right)^2 + \left(\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{f_i} \right)^2 + \left(\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{e_i} \right)^2 + 2\rho_{1,2} \left(\sum_{i=1}^{n_d} \gamma_{i,t}\sigma_{d_i} \right) \left(\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{f_i} \right) \\ &\quad + 2\rho_{2,3} \left(\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{f_i} \right) \left(\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{e_i} \right) + 2\rho_{1,3} \left(\sum_{i=1}^{n_d} \gamma_{i,t}\sigma_{d_i} \right) \left(\sum_{i=1}^{n_f} \beta_{i,t}\sigma_{e_i} \right);\\ u_\xi &= \ln\left[2\mathrm{e}^{u_\pi + \frac{1}{2}\sigma_\pi^2} - 1\right] - \frac{1}{2}\ln\left[1 + \frac{4\mathrm{e}^{2u_\pi + \sigma_\pi^2}(\mathrm{e}^{\sigma_\pi^2} - 1)}{(2\mathrm{e}^{u_\pi + \frac{1}{2}\sigma_\pi^2} - 1)^2}\right]; \qquad \sigma_\xi^2 = \ln\left[1 + \frac{4\mathrm{e}^{2u_\pi + \sigma_\pi^2}(\mathrm{e}^{\sigma_\pi^2} - 1)}{(2\mathrm{e}^{u_\pi + \frac{1}{2}\sigma_\pi^2} - 1)^2}\right].\end{aligned}$$

Eq. (5) is derived in Appendix C. The derivation of u_ξ and σ_ξ^2 is shown in Appendix D.

By means of Eq. (5), one can efficiently obtain the approximation of the VaR capital allocation for an international portfolio. The approximated analytical VaR includes some essential elements such as the volatility of underlying assets, the volatility of exchange rates, the correlation coefficients, the weights of the investment in domestic assets and foreign assets,

Table 1
Summary statistics (January 1, 2004—November 27, 2009).

Statistics	TSMC	MSFT	Exchange rate
Mean	0.000363965	0.000049758	−0.000033361
Standard deviation	0.013228534	0.018771567	0.002783908
Skewness	−0.12503100	−0.264801200	0.124005120
Kurtosis	3.89800000	4.253678000	3.347850000

and the intensity of jumps. Also, Eq. (5) can be reduced to the analytic solution of [16][2] as $\gamma_{i,t} = 1$, $\beta_{i,t} = 0$, $n_d = 1$, $\lambda = 0$, and $\mathrm{d}Y_t = 0$. This case represents that a firm value is only composed of a kind of domestic asset with no jumps. Alternatively, Eq. (5) goes to the closed-form solution of [17][3] as $\gamma_{i,t} = 0$, $\beta_{i,t} = 1$, $n_f = 1$, $\lambda = 0$, and $\mathrm{d}Y_t = 0$, which means that a firm value includes only a kind of foreign asset with no jumps. Also, the presented model can be regarded as the extension of that of [16,17].

3. Measurement of value at risk and numerical analysis

For simplicity, this section considers the long trading positions of an international portfolio with a kind of domestic asset and a kind of foreign asset. From the Taiwanese perspective, the international portfolio includes one domestic-issued stock valued in New Taiwan dollars and one foreign-issued stock valued in US dollars. We then want to know the absolute VaR of the portfolio valued in New Taiwan dollars.

3.1. Source of the data

Assume that the international portfolio includes two specific domestic and foreign stocks which are TSMC and MSFT, respectively. The TSMC stocks, are issued by Taiwan Semiconductor Manufacturing Company Limited and traded in Taiwan; the MSFT stocks are issued by Microsoft and traded in the USA. The daily log returns of TSMC and MSFT stocks are employed. Both of these securities are well-known to institutional and individual investors in the world. The time window length is the period from January 1, 2004 to November 27, 2009, so that the total of the daily log returns of each asset is 1367. All of the samples span two periods, labelled Periods I and II. Period I is from January 1, 2004 to July 31, 2007, during which the subprime mortgage crisis had not yet occurred and the daily log returns totalled 780. Alternatively, Period II is from August 1, 2007 to November 27, 2009 with a total of 587, which is through the subprime mortgage crisis of August 2007.

Table 1 provides some basic statistics on the daily log returns of TSMC, MSFT stocks and exchange rates quoted at the price of one unit of US dollars in New Taiwan dollars from January 1, 2004 to November 27, 2009. Obviously, the distributions of these stock returns and exchange rate returns have heavy tails. The log returns of TSMC and MSFT are negatively skewed. The volatility of TSMC returns is fewer than that of MSFT returns. The volatility of exchange rate returns is the smallest.

3.2. Estimation of model parameters

Before the VaR measurement, it is necessary to estimate a set of model parameters for various samples. Assume that the number of jumps is ten; $u_\pi = 0.05$, $\sigma_\pi^2 = 0.001$, and $\lambda = 0.03$ for Period I, and $u_\pi = 0.055$, $\sigma_\pi^2 = 0.002$, and $\lambda = 0.035$ for Period II. From the data, the sample means and standard deviations of TSMC, MSFT and exchange rates in one day are shown in Table 2. Since one trading day is equivalent to 1/252 year, one can obtain the sample means and variance of these random variables per annum, which are all multiplied by 252 from Panel A in Table 2, respectively. The results are stated in Panel B in Table 2.

From Eqs. (1)–(3), the dynamic processes of log returns of random variables in domestic assets, foreign assets and exchange rates can be derived as Eq. (6), respectively.

$$\begin{aligned}
\mathrm{d}(\ln A_{d_1,t}) &= \left(\mu_{d_1} - \frac{1}{2}\sigma_{d_1}^2 - \lambda v\right)\mathrm{d}t + \sigma_{d_1}\mathrm{d}W_{1,t} + (\pi - 1)\mathrm{d}Y_t, \\
\mathrm{d}(\ln A_{f_1,t}) &= \left(\mu_{f_1} - \frac{1}{2}\sigma_{f_1}^2 - \lambda v\right)\mathrm{d}t + \sigma_{d_1}\mathrm{d}W_{2,t} + (\pi - 1)\mathrm{d}Y_t, \\
\mathrm{d}(\ln e_{1,t}) &= \left(\mu_{e_1} - \frac{1}{2}\sigma_{e_1}^2 - \lambda v\right)\mathrm{d}t + \sigma_{e_1}\mathrm{d}W_{3,t} + (\pi - 1)\mathrm{d}Y_t.
\end{aligned} \tag{6}$$

2 Kupiec [16] shows the absolute VaR as follows: $\mathrm{VaR}_k(0) = A_{d_i,t_0}\left[\exp\left(\left(\mu_{d_i} - \frac{1}{2}\sigma_{d_i}^2\right)T + Z_\alpha\sqrt{\sigma_{d_i}^2 T}\right) - 1\right]$.

3 Chen and Liao [17] derives the absolute VaR of foreign-issued assets as below: $\mathrm{VaR}_c(0) = A_{f_i,t_0}e_{i,t_0}\left\{\exp\left[\left(\mu_{f_i} + \mu_{e_i} - \frac{1}{2}\sigma_{f_i}^2 - \frac{1}{2}\sigma_{e_i}^2\right)T + Z_\alpha\sqrt{(\sigma_{f_i}^2 + \sigma_{e_i}^2 + 2\rho_{2,3}\sigma_{f_i}\sigma_{e_i})T}\right] - 1\right\}$.

Table 2
Sample mean and standard deviation of daily log returns of securities and exchange rates in various periods.

Variables	Period I 2004/1/1–2007/7/31		Period II 2007/8/1–2009/11/27	
	$E[d(\ln H_t)]$	$\sigma(d\ln H_t)$	$E[d(\ln H_t)]$	$\sigma(d\ln H_t)$
Panel A: sample mean and standard deviation of daily log returns				
TSMC	0.000656679	0.015517716	0.000071254	0.010939352
MSFT	0.000064521	0.011705647	0.000034994	0.025837488
NTD/USD	−0.000041458	0.002526791	−0.000025263	0.003041025
Panel B: sample mean and variance of log returns per annum				
Variables	$E[d(\ln H_t)]$	Variance	$E[d(\ln H_t)]$	Variance
TSMC	0.165483108	0.060681476	0.017955756	0.030156694
MSFT	0.016259292	0.034529587	0.008818236	0.168229098
NTD/USD	−0.010447416	0.001608938	−0.006458256	0.002330454

Note that $E[d(\ln H_t)]$ and $\sigma(d\ln H_t)$ represent the sample means and standard deviations of daily log returns of domestic assets, foreign assets, and exchange rates for all $H_t = A_{d_1,t}, A_{f_1,t}, e_{1,t}$, respectively. Panel B displays the sample means and variances of domestic assets, foreign assets, and exchange rates per annum all multiplied by 252 from Panel A, respectively.

Table 3
Parameter estimation of dynamic processes of asset returns and exchange rate returns in various periods.

Security and exchange rate	Period I 2004/1/1–2007/7/31		Period II 2007/8/1–2009/11/27	
	μ_i	σ_i	μ_i	σ_i
TSMC	0.1959	0.2463	0.0331	0.1735
MSFT	0.1858	0.1799	0.0929	0.4102
NTD/USD	−0.0096	0.0397	−0.0052	0.0475

Note that assume the number of jumps is ten. Given $u_\pi = 0.05$, σ_π^2=0.001, and $\lambda = 0.03$ for Period I, and $u_\pi = 0.055$, $\sigma_\pi^2 = 0.002$, and $\lambda = 0.035$ for Period II, μ_i and σ_i demonstrate the estimation of drift terms and volatilities of asset returns and foreign exchange returns for $i = d_1, f_1$, and e_1, respectively.

Table 4
Estimation of the correlation coefficients in various periods.

Correlation coefficients	Period I 2004/1/1–2007/7/31	Period II 2007/8/1–2009/11/27
$\rho_{1,2}$	0.155207	0.206361
$\rho_{2,3}$	0.052375	0.081776
$\rho_{1,3}$	0.023758	0.043939

Note that $\rho_{1,2}$, $\rho_{2,3}$, and $\rho_{1,3}$ denote the correlation coefficients between domestic assets (TSMC) and foreign assets (MSFT), foreign assets (MSFT) and exchange rates (NTD/USD), domestic assets (TSMC) and exchange rates (NTD/USD), respectively.

Furthermore, the estimated results of μ_{d_1}, μ_{f_1} and μ_{e_1} can be determined as $E[d(\ln H_t)] + \frac{1}{2}\sigma_i^2 + \lambda v$ with $v = \exp\left[u_\pi + \frac{1}{2}\sigma_\pi^2\right] - 1$ for all $H_t = A_{d_1,t}, e_{1,t}$, and $A_{f_1,t}$, and $i = d_1, f_1, e_1$, respectively.

Similarly, σ_{d_1}, σ_{f_1}, and σ_{e_1} can be respectively estimated through the variances of Eq. (6) because $\mathrm{Var}\left[\mathrm{d}(\ln A_{d_1,t})\right] = \sigma_{d_1}^2 \mathrm{d}t + \sigma_\pi^2\lambda$, $\mathrm{Var}\left[\mathrm{d}(\ln A_{f_1,t})\right] = \sigma_{f_1}^2 \mathrm{d}t + \sigma_\pi^2\lambda$, and $\mathrm{Var}\left[\mathrm{d}(\ln A_{e_1,t})\right] = \sigma_{e_1}^2 \mathrm{d}t + \sigma_\pi^2\lambda$. Finally, the estimated results of $\mu_{d_1}, \mu_{f_1}, \mu_{e_1}, \sigma_{d_1}, \sigma_{f_1}$ and σ_{e_1} are presented in Table 3.

In addition, Table 4 reports the estimations of the correlation coefficients between each asset and exchange rates in various periods.

3.3. Calculation of VaR

After the estimation of model parameters, we can quickly obtain a one-day VaR at a 0.01 significance level for the international portfolio on TSMC and MSFT through Eq. (5). These results are summarized in Tables 5 and 6 given that the jump number is 10, $Z_{0.01} = -2.33$ at a 0.01 quantile, $T = 1/252$ and $V_0 = 1$ (initial investment); $u_\pi = 0.05$, $\sigma_\pi^2 = 0.001$, and $\lambda = 0.03$ for Period I, and $u_\pi = 0.055$, $\sigma_\pi^2 = 0.002$, and $\lambda = 0.035$ for Period II. Clearly, there exists a common phenomenon—the maximum losses of initial investment of 1 New Taiwan dollar in Period I are fewer than those in Period II, as the weights of foreign assets and correlation coefficients change. This indicates that it is necessary for a firm to maintain a sufficient capital amount in order to prevent default risk during the subprime mortgage crisis period. In addition, it can decrease the losses of the portfolio for investors to decline weights of foreign assets during the subprime mortgage crisis period.

Table 5
Model accuracy using backtesting in terms of in-sample fitting for alternative correlation coefficients.

ρ_{13}	Period I 2004/1/1–2007/7/31				Period II 2007/8/1–2009/11/27			
	Number of samples	VaR	Number of exceptions	LR_{uc}	Number of samples	VaR	Number of exceptions	LR_{uc}
	Panel A: correlation coefficient between domestic assets and foreign exchange rates changes							
0.2	780	−0.05591	5	1.1633	587	−0.07162	6	0.0029
0.4	780	−0.05984	5	1.1633	587	−0.07965	5	0.1371
0.6	780	−0.06126	4	2.2760	587	−0.08197	5	0.1371
0.8	780	−0.06298	4	2.2760	587	−0.08869	4	0.6775
ρ_{23}	Panel B: correlation coefficient between foreign assets and foreign exchange rates changes							
0.2	780	−0.05293	5	1.1633	587	−0.06891	5	0.1371
0.4	780	−0.05585	4	2.2760	587	−0.07133	4	0.6775
0.6	780	−0.05897	4	2.2760	587	−0.07964	4	0.6775
0.8	780	−0.06179	3	3.8967*	587	−0.08157	3	1.7267

Note that this table displays backtesting in terms of in-sample fitting for alternative weights of domestic assets. The critical value is 3.84 at a significant level of 5%. The VaRs are the maximum losses of the initial investment of 1 New Taiwan dollar (NTD) over a one-day horizon.
* denotes the significance at a 5% level.

Table 6
Model accuracy using backtesting in terms of in-sample fitting for alternative weights of domestic assets.

Weights of domestic assets ($\gamma_{1,t}$)	Period I 2004/1/1–2007/7/31				Period II 2007/8/1–2009/11/27			
	Number of samples	VaR	Number of exceptions	LR_{uc}	Number of samples	VaR	Number of exceptions	LR_{uc}
0	780	−0.0533	6	0.4558	587	−0.0804	0	11.7990*
0.1	780	−0.0481	6	0.4558	587	−0.0728	1	6.2410*
0.2	780	−0.0436	6	0.4558	587	−0.0659	2	3.4589
0.3	780	−0.0401	4	2.2760	587	−0.0598	2	3.4589
0.4	780	−0.0377	4	2.2760	587	−0.0549	2	3.4589
0.5	780	−0.0366	7	0.0858	587	−0.0516	3	1.7267
0.6	780	−0.0371	7	0.0858	587	−0.0503	2	3.4589
0.7	780	−0.0389	9	0.1777	587	−0.0501	2	3.4589
0.8	780	−0.0419	9	0.1777	587	−0.0502	3	1.7267
0.9	780	−0.0458	9	0.1777	587	−0.0487	4	0.6775
1	780	−0.0504	9	0.1777	587	−0.0547	6	0.0029

Note that this table displays backtesting in terms of in-sample fitting for alternative weights of domestic assets. The critical value is 3.84 at a significant level of 5%. The VaRs are the maximum losses of the initial investment of 1 New Taiwan dollar (NTD) over a one-day horizon.
* denotes the significance at a 5% level.

Alternatively, an ordinary Monte Carlo simulation approach is employed to calculate the VaRs of the international portfolio under no specific assumptions about the distribution of risk factors. The Monte Carlo simulation is based on 4 time steps (representing 4 quarter in a period) and 50 000 trials. First, the following variables are obtained through Eq. (6) for simulation l.

$$\begin{aligned}
A_{d_1,t} &= A_{d_1,0}\exp\left[\left(\mu_{d_1}-\frac{1}{2}\sigma_{d_1}^2-\lambda v\right)\Delta t+\sigma_{d_1}\sqrt{\Delta t}\varepsilon_{d,t}\right]Y(N),\\
A_{f_1,t} &= A_{f_1,0}\exp\left[\left(\mu_{f_1}-\frac{1}{2}\sigma_{f_1}^2-\lambda v\right)\Delta t+\sigma_{f_1}\sqrt{\Delta t}\varepsilon_{f,t}\right]Y(N),\\
e_{1,t} &= e_{1,0}\exp\left[\left(\mu_{e_1}-\frac{1}{2}\sigma_{e_1}^2-\lambda v\right)\Delta t+\sigma_{e_1}\sqrt{\Delta t}\varepsilon_{e,t}\right]Y(N),
\end{aligned} \tag{7}$$

where $Y(N) = \prod_{a=1}^{N} Y_a$, and $\{Y_a\}$ is an independent Poisson series. $\varepsilon_{d,t}, \varepsilon_{f,t}$, and $\varepsilon_{e,t}$ independently follow normal distribution of zero mean and 1 variance. Furthermore, we use the Cholesky decomposition to obtain the correlation matrix among $\varepsilon_{d,t}, \varepsilon_{f,t}$, and $\varepsilon_{e,t}$. Then one can make $\varepsilon_{d,t}, \varepsilon_{f,t}$, and $\varepsilon_{e,t}$ be correlated through the correlation matrix. Next, $\mathrm{VaR}_j(0)$ is obtained from Eq. (5) for simulation l. We repeat the previous procedure 50 000 times and sum the $\mathrm{VaR}_j(0)$ for all $l = 1, 2, \ldots, 50\,000$. Finally, the mean of the sum of $\mathrm{VaR}_j(0)$ is gained, and we can regard it as the amount of VaR in terms of ordinary Monte Carlo simulations as illustrated in Table 8.

Table 8 consistently demonstrates that the losses valued by the analytical VaR are higher than those by the Monte Carlo simulation approach in various domestic weights during both Period I and Period II. If financial managers adopt the historical simulation approach to evaluate financial risk, the firm's financial ratio (such as ROE) is better. However, the default

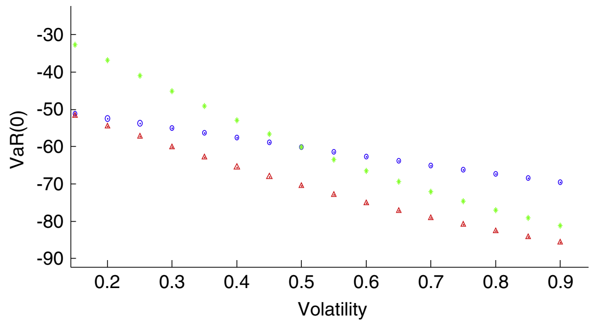

Fig. 1. The impact of volatility on VaR. Note that the symbols "*o*", "∗" and "∧" represent the impact of volatilities of domestic assets, foreign assets and exchange rates on absolute VaRs, respectively.

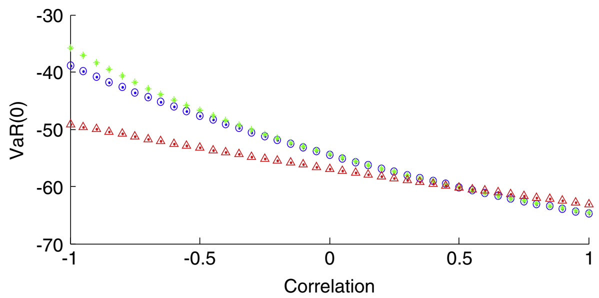

Fig. 2. The impact of correlation coefficients on VaR. Note that the symbols "*o*", "∗" and "∧" represent the impact of correlation coefficients, $\rho_{1,2}$, $\rho_{2,3}$, $\rho_{1,3}$ on absolute VaRs, respectively.

probability of the firm may increase on the account of a shortage of sufficient capital requirement. Hence the conservative policy of the analytical VaR model would be suitable for financial institutions to control market risk.

3.4. Numerical analysis

Based on the estimation of the model parameters shown in Tables 3 and 4, this section provides the sensitivity analyses of the impacts of important parameters on VaR capital in terms of comparative statics. We start by assuming that (i) the value of a firm is made up of a line of a domestic asset and a foreign asset, and the exchange rate is the ratio of the domestic currency to the foreign currency; (ii) the initial value of an international portfolio is $100; (iii) the critical value is −2.33 at a given α of 0.01, and the investment horizon is one year ($T = 1$); (iv) $\gamma_{1,t} = 0.3$ and $\beta_{1,t} = 0.7$.

According to Eq. (5), the effects of volatilities, correlation coefficients, and the intensities of jumps on the absolute VaR capital allocation are shown in Figs. 1–3, respectively. There is one common phenomenon exhibited in these figures: the loss amount increases monotonically as volatilities, correlation coefficients and the intensities of jumps rise. As shown in Figs. 1 and 2, the sensitivities of the volatility of foreign assets and the correlation coefficient between foreign assets and exchange rates are higher than those of the others. Additionally, Fig. 4 illustrates the relationship between the VaR and the weights of hump-shaped domestic assets shapes in hump. Also, the loss amount declines as the weights of foreign assets rise at around 0.5.

4. Evaluation of model accuracy

Backtesting is a widely used method of evaluating VaR accuracy. Moreover, we will compare the accuracy of the analytical VaR derived from Eq. (5) with that of the Monte Carlo simulation in terms of backtesting criterion for the international portfolio on the TSMC and MSFT stocks through in-sample and out-of-sample fitting.

The usual backtesting techniques consider the number of violations at which the losses are larger than VaR. The proportion of times should be equal to one minus the VaR confidence level; in other words, the model should provide the correct unconditional coverage. In order to test the null hypothesis that the unconditional coverage equals the significant level, Kupiec [18] presents a likelihood ratio statistic. Given a VaR at the 1% level left-tail over daily horizon for a total of D, one can count how many times the actual loss exceeds one day's VaR. Define d as the number of exceptions and d/D as the

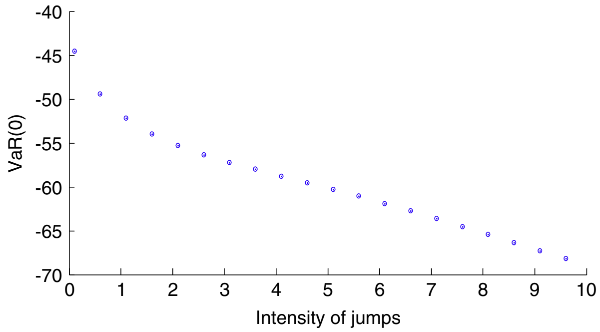

Fig. 3. The impact of the intensity of jumps on VaR.

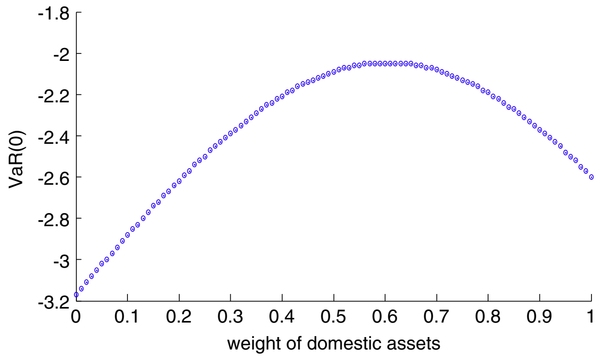

Fig. 4. The impact of weights of domestic assets on VaR.

exception rate. The null hypothesis is that a given confidence level for losses is the true probability. Kupiec [18] approximates 95% confidence regions, denoted by q for the test. The unconditional coverage is defined by the log-likelihood ratio:

$$LR_{uc} = -2\ln\left[(1-q)^{D-d}q^d\right] + 2\ln\left\{\left[1-\frac{d}{D}\right]^{D-d}\left(\frac{d}{D}\right)^d\right\}. \tag{8}$$

The LR_{uc} statistic has a chi-square distribution with one degree of freedom. One would reject the null hypothesis if $LR_{uc} > 3.84$ at a 95% confidence level. The test procedure described above is called backtesting.

Assume that the jump number is 10, $Z_{0.01} = -2.33$ at a 0.01 quantile, T=1/252 and $V_0 = 1$ (initial investment); $u_\pi = 0.05, \sigma_\pi^2 = 0.001$, and $\lambda = 0.03$ for Period I, and $u_\pi = 0.055, \sigma_\pi^2 = 0.002$, and $\lambda = 0.035$ for Period II. In in-sample fitting, the time window length is the period from January 1, 2004 to November 27, 2009, which is broken into two periods, labelled Period I and Period II. Period I is from January 1, 2004 to July 31, 2007, during which the subprime mortgage crisis had not yet occurred; Period II is from August 1, 2007 to November 27, 2009. Tables 5 and 6 demonstrate that the null hypothesis for the international portfolio can almost not be rejected at a significance level of 5% in Period I and Period II as domestic weights and correlation coefficients change.

In out-of-sample approach, we also split the data sample into two parts. The first part is used to estimate the model from January 1, 2004 to July 31, 2007 (estimated in Period I). The second part is used to forecast VaRs from August 1, 2007 to November 27, 2009 (forecasted in Period II). The statistics of the unconditional coverage are illustrated in Table 7 showing how one fails to reject the null hypothesis that a given confidence level for losses is the true probability in terms of out-of-sample fitting for alternative weights of domestic assets.

In addition, Table 8 records the computation time and VaRs by both Monte Carlo simulation and the approximated solution from Eq. (5). Obviously, not only are the VaRs from Eq. (5) nearly close to those of the Monte Carlo simulation, but also the VaR solution of Eq. (5) can efficiently save computational cost.

To summarize, the VaR model presented by this paper can be used to accurately calculate VaR for an international portfolio based on in-sample and out-of-sample approaches.

5. Conclusion

One advantage of VaR is that it is an intuitively appealing measure of risk that can be easily conveyed to a firm's senior manager. The measure most commonly used assumes that the probability distribution of daily asset returns is normal.

Table 7
Model accuracy using backtesting in terms of out-of-sample fitting for alternative weights of domestic assets.

Weights of domestic assets ($\gamma_{1,t}$)	Number of samples	VaR	Number of exceptions	LR_{uc}
0	587	−0.0533	16	12.0051
0.1	587	−0.0481	15	10.0301*
0.2	587	−0.0436	11	3.6023
0.3	587	−0.0401	11	3.6023
0.4	587	−0.0377	10	2.4240
0.5	587	−0.0366	11	3.6023
0.6	587	−0.0371	9	1.4490
0.7	587	−0.0389	8	0.7012
0.8	587	−0.0419	9	1.4490
0.9	587	−0.0458	9	1.4490
1	587	−0.0504	8	0.7012

Note that this table displays backtesting in terms of out-of-sample fitting for alternative weights of domestic assets. The data samples are split into two parts. The 1-day VaRs are estimated from January 1, 2004 to July 31, 2007 (Period I). The second part is used to forecast VaRs from August 1, 2007 to November 27, 2009 (Period II). The number of exceptions indicates the times that the VaRs are exceeded in Period II. The critical value is 3.84 at a significant level of 5%.
* denotes the significance at a 5% level.

Table 8
Comparison with Monte Carlo simulation for alternative weights of domestic assets.

Domestic weights	Period I 2004/1/1–2007/7/31							Period II 2007/8/1–2009/11/27						
	VaR_1	CT_1	VaR_2	SE	CT_2	$LR_{1,uc}$	$LR_{2,uc}$	VaR_1	CT_1	VaR_2	SE	CT_2	$LR_{1,uc}$	$LR_{2,uc}$
0	−0.0533	3″	−0.0535	0.00032	4′11″	0.4558	0.4558	−0.0804	3″	−0.0803	0.00029	4′12″	11.7990*	11.7990*
0.1	−0.0481	3″	−0.0484	0.00031	4′12″	0.4558	0.4558	−0.0728	3″	−0.0725	0.00027	4′12″	6.2410*	6.2410*
0.2	−0.0436	3″	−0.0441	0.00029	4′11″	0.4558	0.4558	−0.0659	3″	−0.0655	0.00031	4′12″	3.4589	3.4589
0.3	−0.0401	3″	−0.0411	0.00026	4′12″	2.2760	2.2760	−0.0598	3″	−0.0599	0.00026	4′12″	3.4589	3.4589
0.4	−0.0377	3″	−0.0380	0.00036	4′12″	2.2760	2.2760	−0.0549	3″	−0.0551	0.00032	4′12″	3.4589	3.4589
0.5	−0.0366	3″	−0.0369	0.00029	4′11″	0.0858	0.0858	−0.0516	3″	−0.0518	0.00031	4′11″	1.7267	1.7267
0.6	−0.0371	3″	−0.0380	0.00025	4′11″	0.0858	0.0858	−0.0503	3″	−0.0508	0.00037	4′11″	3.4589	3.4589
0.7	−0.0389	3″	−0.0391	0.00041	4′11″	0.1777	0.1777	−0.0501	3″	−0.0511	0.00043	4′11″	3.4589	3.4589
0.8	−0.0419	3″	−0.0421	0.00038	4′11″	0.1777	0.1777	−0.0502	3″	−0.0508	0.00036	4′12″	1.7267	1.7267
0.9	−0.0458	3″	−0.0460	0.00033	4′12″	0.1777	0.1777	−0.0487	3″	−0.0489	0.00035	4′12″	0.6775	0.6775
1	−0.0504	3″	−0.0514	0.00045	4′12″	0.1777	0.1777	−0.0547	3″	−0.0551	0.00041	4′12″	0.0029	0.0029

Note that VaR_1 and VaR_2 stand for one-day VaRs at a 1% quantile, which are measured from Eq. (5) and the Monte Carlo simulation, respectively. CT_1 and CT_2 represent the computation time based on Eq. (5) and the Monte Carlo simulation, respectively. $x'y''$ indicates that a simulation takes x minutes and y seconds. SE is the standard errors of the Monte Carlo estimates. The Monte Carlo simulation is based on 4 time steps and 50 000 trials. $LR_{1,uc}$ and $LR_{2,uc}$ stand for the unconditional coverage rates of the analytical VaR and Monte Carlo simulation in terms of in-sample fit, respectively. The critical value is 3.84 at a significant level of 5%.
* denotes the significance at a 5% level.

However, this assumption is far from conditions in the actual world. This paper provides a mixed Poisson-jump model for an international portfolio to manage market risk, in particular the subprime mortgage crisis of August 2007. Differing from past studies whose portfolios were only valued in one currency, this model considers portfolios not only with jumps but also with exchange rate risk. It is vital for investors to consider exchange rate risk in highly integrated global financial markets.

Additionally, through backtesting criterion, the finding is that the model is more capable of accurately reflecting the loss probability of 1% in terms of in-sample and out-of-sample fitting. Hence, the proposed method in this paper is a more efficient way in the presence of asymmetric and fat-tail portfolio returns during periods of financial turbulence.

Appendix A. The proof of Proposition 1

Let $\sum_{i=1}^{n_f} g_i e_{i,t} A_{f_i,t}$ with $g_{i,t}$ foreign asset shares. Conditional on self-financing strategy and by means of Ito's lemma, one can obtain

$$\frac{\mathrm{d}X_{i,t}}{X_{i,t}} = \sum_{i=1}^{n_f} g_i \left(\frac{\mathrm{d}A_{f_i,t}}{A_{f_i,t}} + \frac{\mathrm{d}e_{i,t}}{e_{i,t}} + \frac{\mathrm{d}A_{f_i,t}}{A_{f_i,t}} \cdot \frac{\mathrm{d}e_{i,t}}{e_{i,t}} \right). \tag{A.1}$$

Substituting the dynamic processes of foreign asset returns and exchange rates shown in Eqs. (2) and (3) into (A.1), Eq. (A.1) can be expressed as

$$\frac{\mathrm{d}X_{i,t}}{X_{i,t}} = \sum_{i=1}^{n_f} g_i \left[(\mu_{f_i} + \mu_{e_i} - 2\lambda \upsilon + \rho_{2,3}\sigma_{f_i}\sigma_{e_i})\mathrm{d}t + \sigma_{f_i}\mathrm{d}W_{2,t} + \sigma_{e_i}\mathrm{d}W_{3,t} + 2(\pi - 1)\mathrm{d}Y_t \right].$$

Appendix B. The proof of Proposition 2

Suppose $V_t=\sum_{i=1}^{n_d} m_i A_{d_i,t}+\sum_{i=1}^{n_f} g_i e_{i,t} A_{f_i,t}$. Conditional on assumption (vi) and using Ito's lemma, one can obtain

$$\frac{\mathrm{d}V_t}{V_t}=\sum_{i=1}^{n_d}\gamma_{i,t}\frac{\mathrm{d}A_{d_i,t}}{A_{A_{d_i},t}}+\sum_{i=1}^{n_f}\beta_{i,t}\frac{\mathrm{d}X_{i,t}}{X_{i,t}}, \tag{B.1}$$

with $\gamma_{i,t}=\frac{m_i A_{d_i,t}}{V_t}$, and $\beta_{i,t}=\frac{g_i e_{i,t} A_{f_i,t}}{V_t}$. Substituting Proposition 1, and Eq. (1) into (B.1), the result is as follows:

$$\begin{aligned}\frac{\mathrm{d}V_t}{V_t}&=\left[\sum_{i=1}^{n_d}\gamma_{i,t}(\mu_{d_i}-\lambda v)+\sum_{i=1}^{n_f}\beta_{i,t}(\mu_{e_i}+\mu_{f_i}-2\lambda v+\rho_{2,3}\sigma_{f_i}\sigma_{e_i})\right]\mathrm{d}t\\&\quad+\sum_{i=1}^{n_d}\gamma_{i,t}\sigma_{d_i}\mathrm{d}W_{1,t}+\left[\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{f_i}\right]\mathrm{d}W_{2,t}+\left[\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{e_i}\right]\mathrm{d}W_{3,t}+2(\pi-1)\mathrm{d}Y.\end{aligned}$$

Appendix C. The derivation of Eq. (5)

Given a confidence level of α, VaR can be expressed as

$$P_r(V_T\le V_\alpha)=\alpha. \tag{C.1}$$

Based on the absolute VaR being denoted by $\mathrm{VaR}(0)\equiv V_\alpha-V_0$, Eq. (C.1) can be transformed into

$$P_r(V_T\le V_0+\mathrm{VaR}(0))=\alpha. \tag{C.2}$$

Let $\sigma_t\mathrm{d}W_t\equiv\sum_{i=1}^{n_d}\gamma_{i,t}\sigma_{d_i}\mathrm{d}W_{1,t}+\left[\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{f_i}\right]\mathrm{d}W_{2,t}+\left[\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{e_i}\right]\mathrm{d}W_{3,t}$. From Proposition 2, one can obtain

$$\ln V_T|Y_T=k\sim N\left(\ln V_0+\left(\mu_t-\frac{1}{2}\sigma_t^2\right)T+ku_\xi,\,\sigma_t^2T+k\sigma_\xi^2\right), \tag{C.3}$$

in which k stands for the number of jumps and satisfies $k=0,1,\dots,\infty$; $N(.)$ represents a normal distribution; u_ξ and σ_ξ^2 denote the expected value and the variance of the nature logarithm of $2\pi-1$, respectively;

$$\mu_t=\sum_{i=1}^{n_d}\gamma_{i,t}(\mu_{d_i}-\lambda v)+\sum_{i=1}^{n_f}\beta_{i,t}(\mu_{e_i}+\mu_{f_i}-2\lambda v+\rho_{2,3}\sigma_{f_i}\sigma_{e_i});$$

$$\begin{aligned}\sigma_t^2&=\left(\sum_{i=1}^{n_d}\gamma_{i,t}\sigma_{d_i}\right)^2+\left(\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{f_i}\right)^2+\left(\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{e_i}\right)^2+2\rho_{1,2}\left(\sum_{i=1}^{n_d}\gamma_{i,t}\sigma_{d_i}\right)\left(\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{f_i}\right)\\&\quad+2\rho_{2,3}\left(\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{f_i}\right)\left(\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{e_i}\right)+2\rho_{1,3}\left(\sum_{i=1}^{n_d}\gamma_{i,t}\sigma_{d_i}\right)\left(\sum_{i=1}^{n_f}\beta_{i,t}\sigma_{e_i}\right).\end{aligned}$$

Assume $\sigma_B\mathrm{d}B_t\equiv\sigma_t\mathrm{d}W_t+\ln(2\pi-1)\mathrm{d}k$. Thus, Eq. (C.2) becomes

$$\sum_{k=0}^{\infty}P_r\left(Y_T=k\right)P_r\left(\sigma_t W_T+k\ln(2\pi-1)\le\ln\left[V_0+\mathrm{VaR}(0)\right]-\ln V_0-\left(\mu_t-\frac{1}{2}\sigma_t^2\right)T|Y_T=k\right)=\alpha. \tag{C.4}$$

Because jumps follow Poisson distribution, Eq. (C.4) can be easily written through (C.3) as (C.5)

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\exp\left[-\lambda T\right]\left[\lambda T\right]^k}{k!}P_r\left(\frac{\sigma_t W_T+k\ln(2\pi-1)-ku_\xi}{\sqrt{\sigma_t^2T+k\sigma_\xi^2}}\le\frac{\ln\left[V_0+\mathrm{VaR}(0)\right]-\ln V_0-\left(\mu_t-\frac{1}{2}\sigma_t^2\right)T-ku_\xi}{\sqrt{\sigma_t^2T+k\sigma_\xi^2}}\right)\\&=\alpha.\end{aligned} \tag{C.5}$$

Consequently, Eq. (5) can be proved:

$$\sum_{k=0}^{\infty}\frac{\exp\left[-\lambda T\right]\left[\lambda T\right]^k}{k!}\Phi\left(\frac{\ln\left[V_0+\mathrm{VaR}(0)\right]-\ln V_0-\left(\mu_t-\frac{1}{2}\sigma_t^2\right)T-ku_\xi}{\sqrt{\sigma_t^2T+k\sigma_\xi^2}}\right)=\alpha.$$

Appendix D. The derivation of the expected value and variance of $\ln(2\pi - 1)$

Given π follows a lognormal distribution with parameters u_π and σ_π^2, the probability distribution of $2\pi - 1$ is also lognormal. Let $u_\xi \equiv E[\ln(2\pi - 1)]$, and $\sigma_\pi^2 \equiv \mathrm{Var}[\ln(2\pi - 1)]$. Because $E[\pi] = e^{u_\pi + \frac{1}{2}\sigma_\pi^2}$ and $\mathrm{Var}[\pi] = e^{2u_\pi + \sigma_\pi^2}(e^{\sigma_\pi^2} - 1)$, the expected value and variance of $2\pi - 1$ are obtained as follows:

$$E[2\pi - 1] = 2e^{u_\pi + \frac{1}{2}\sigma_\pi^2} - 1 = e^{u_\xi + \frac{1}{2}\sigma_\xi^2}, \tag{D.1}$$

$$\mathrm{Var}[2\pi - 1] = 4e^{2u_\pi + \sigma_\pi^2}(e^{\sigma_\pi^2} - 1) = e^{2u_\xi + \sigma_\xi^2}(e^{\sigma_\xi^2} - 1). \tag{D.2}$$

From Eqs. (D.1) and (D.2), u_ξ and σ_π^2 can be solved as follows:

$$u_\xi = \ln\left[2e^{u_\pi + \frac{1}{2}\sigma_\pi^2} - 1\right] - \frac{1}{2}\ln\left[1 + \frac{4e^{2u_\pi + \sigma_\pi^2}(e^{\sigma_\pi^2} - 1)}{\left(2e^{u_\pi + \frac{1}{2}\sigma_\pi^2} - 1\right)^2}\right], \quad \text{and} \quad \sigma_\xi^2 = \ln\left[1 + \frac{4e^{2u_\pi + \sigma_\pi^2}(e^{\sigma_\pi^2} - 1)}{\left(2e^{u_\pi + \frac{1}{2}\sigma_\pi^2} - 1\right)^2}\right].$$

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